A NEW SOFT SET OPERATION: COMPLEMENTARY SOFT BINARY PIECEWISE UNION (∪) OPERATION

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1. INTRODUCTION

We are unable to properly use traditional approaches to address problems in many domains, including economics, environmental and health sciences, and engineering, due to the presence of certain types of uncertainty. The Probability Theory, Fuzzy Set Theory, and Interval Mathematics are three well-known fundamental theories that we might take into consideration as a mathematical instrument to cope with uncertainties. Molodtsov (Molodtsov, 1999) presented Soft Set Theory as a mathematical technique to address these uncertainties, since each of theories has its own drawbacks, this approach has limitations as well. Since then, several domains, such as information systems, decision-making, optimization theory, game theory, operations research, measurement theory, and others have used theory.

In terms of soft set operations, the first contributions were published in (Maji et al., 2003; Pei and Miao, 2005). The introduction and analysis of a number of soft set operations (including restricted and extended soft set operations) followed in (Ali et al., 2009). In the study, the fundamental characteristics of soft set operations were covered, and the relationships between them were presented (Sezgin and Atagın, 2011). Additionally, they explored the concept of restricted symmetric difference of soft sets and described it. Extended symmetric difference of soft sets was defined and its properties were examined in (Stojanovic, 2021). Extended difference of soft sets is a novel soft set operation that was introduced in (Sezgin et. al. 2019). When we look at the research, we can see that restricted soft set operations and extended soft set operations are the two primary categories under which soft set theory’s operations fall.

A novel idea in set theory known as the inclusive complement and exclusive complement of sets were suggested, and the connections between them were examined in (Çağman, 2021). Some new characteristics of sets were defined as a result of this study’s inspiration (Sezgin et. al. 2023a). Additionally, they used these complements to soft set theory and established several new restricted and extended soft set operations (Sezgin and Aybek, 2023a). By altering the form of extended soft set operations using the complement at the first and second row of the piecewise function of extended soft set operations, recent studies defined a new type of extended operation and thoroughly examined their fundamental characteristics (Sezgin and Demirci, 2023a; Sezgin and Saralioğlu, 2023a; Sezgin and Akbulut, 2023). Additionally, a new class of soft difference operations was developed by (Eren and Çalışıcı, 2019). Others defined several new soft set operations, known as soft binary piecewise operations, and thoroughly examined their fundamental features (Sezgin and Yavuz, 2023a). Additionally, by describing a new kind of soft binary piecewise operation, the studies continued their work on soft set operations. Utilizing the complement in the first row of the soft binary piecewise operations, they modified the soft binary operation’s form (Sezgin and Demirci, 2023b; Sezgin and Saralioğlu, 2023b; Sezgin and Atagın, 2023; Sezgin and Aybek, 2023c; Sezgin et al., 2023b; Sezgin and Yavuz, 2023b; and Sezgin and Çağman, 2023).

The purpose of this work is to describe a brand-new soft set operation that we refer to as “complementary soft binary piecewise union operation” in order to add to the body of literature on soft set theory. The operation’s definition and an example are provided for this purpose, and the detailed analysis of this new operation’s algebraic properties, including closure, association, unit, inverse element, and abelian property, are examined. The distributions of complementary soft binary piecewise union operation over extended soft set operations, complementary extended soft set operations, soft binary piecewise operations, operations, and restricted soft set operations are specifically targeted in order to contribute to the literature on soft sets.

ABSTRACT

Molodtsov’s soft set theory has been used in various disciplines both theoretically and practically. It is an effective mathematical tool for dealing with uncertainty. Since its debut, numerous types of soft set operations have been presented and applied. In this study, we analyze the fundamental algebraic features of a novel soft set operation which we call complementary soft binary piecewise union operation. Additionally, by examining the distribution of complementary soft binary piecewise union operation over all other soft set operations’ type, it aims to contribute to the literature on soft sets by revealing relationships between this new soft set operation and other types of soft set operations. Moreover, we prove that the set of all the soft sets with a fixed parameter set together with the complementary soft binary piecewise union operation and the soft binary piecewise intersection operation is a zero-symmetric near-semiring.

KEYWORDS

Soft Sets, Soft Set Operations, Conditional Complements, Near-semirings

2. Preliminaries

Definition 2.1 Let $U$ be the universal set, $E$ be the parameter set, $P(U)$ be the power set of $U$ and $A \subseteq E$. A pair $(F, A)$ is called a soft set over $U$ where $F$ is a set-valued function such that: $F: A \rightarrow P(U)$ (Molodtsov, 1999).

Throughout this paper, the set of all soft sets over $U$ is designated by $S(U)$. Let $A$ be a fixed subset of $E$ and $S(U)$ be the collection of all soft sets over $U$ with the fixed parameters set $A$. Clearly $S(U)$ is a subset of $S(U)$.

Definition 2.2 $(P, D)$ is called a relative null soft set (with respect to the parameter set $D$), denoted by $\varnothing_\sigma$, if $P(t) = \varnothing$ for all $t \in D$ and $(P, D)$ is called a relative whole soft set (with respect to the parameter set $D$), denoted by $U_\sigma$, if $P(t) = U$ for all $t \in D$. The relative whole soft set $U_\sigma$ with respect to the universe set of parameters $E$ is called the absolute soft set over $U$ (Ali et al., 2009).

Definition 2.3 For two soft sets $(P, D)$ and $(R, J)$, we say that $(P, D)$ is a soft subset of $(R, J)$ and it is denoted by $(P, D) \subseteq (R, J)$, if $D \subseteq J$ and $P(t) \subseteq R(t)$, $\forall t \in D$. Two soft sets $(P, D)$ and $(R, J)$ are said to be soft equal if $(P, D)$ is a soft subset of $(R, J)$ and $(R, J)$ is a soft subset of $(P, D)$ (Pei and Mao, 2005).

Definition 2.4 The relative complement of a soft set $(P, D)$, denoted by $(P, D)^\prime = (P^\prime, A)$, is defined if $P^\prime(t) = D \setminus P(t)$ for all $t \in D$. The following operations are considered:

1. The restricted intersection operation, $\cap\sigma$: $(P, D)^\prime \cap\sigma (R, J)^\prime$, denoted by $(P \cap\sigma R, D \cap J)$.
2. The restricted union operation, $\cup\sigma$: $(P, D)^\prime \cup\sigma (R, J)^\prime$, denoted by $(P \cup\sigma R, D \cup J)$.
3. The relative difference operation, $\setminus\sigma$: $(P, D)^\prime \setminus\sigma (R, J)^\prime$, denoted by $(P \setminus\sigma R, D \setminus J)$.

Definition 2.5 Let $(P, D)$ and $(R, J)$ be soft sets over $U$. The restricted $\cap\sigma$ operation of $(P, D)$ and $(R, J)$ is the soft set $(S, F)$, denoted by $(P, D)^\prime \cap\sigma (R, J)$, where $F = D \cap J \neq \varnothing$ and $\forall t \in F, S(t) = P(t) \cap R(t)$ (Ali et al., 2009). From now on, $U = \cap\sigma D = \cap\sigma D \cap D$ will be denoted by $P(D)$ as the sake of designation.

In the inclusive complement and exclusive complement of sets, two conditional complements of sets, were defined. We represent their complements as + and $\theta$, respectively, to make the illustration easier (Çağman, 2021). The following definitions of these complements, which are binary operations: Assume that $U$ has two subsets, $D$ and $J$. The formulas for the $+\theta$-inclusive complement of $D$ and $-\theta$-exclusive complement are $D + \theta = D \cup J$ and $D - \theta = D \cap J$, respectively. Here $U$ refers to a universe, $D$ is the complement of $D$ over $U$. For more information, we refer to (Çağman, 2021).

The relationships between these two complements were thoroughly investigated, and they also developed new three complements as binary operations of sets, such as the ones listed below: Assume that $U$ has two subsets, $D$ and $J$. The formulas for the $+\theta$-inclusive complement of $D$ and $-\theta$-exclusive complement are $D + \theta = D \cup J$ and $D - \theta = D \cap J$, respectively. Here $U$ refers to a universe, $D$ is the complement of $D$ over $U$. The standard examples of near-semirings are typically of the form $(\mathbb{F}, +, \cdot)$, equipped with composition of mappings, pointwise addition of mappings, and the zero function. Subsets of $\mathbb{F}$ closed under the operations provide further examples of near-semirings.

3. Complementary Soft Binary Piecewise Union ($\cup$) Operation and Its Properties

Definition 3.1. Let $(F, A)$ and $(G, B)$ be soft sets over $U$. The complementary soft binary piecewise union ($\cup$) operation of $(F, A)$ and $(G, B)$ is the soft set $(H, A)$, denoted by $(F, A) \cup (G, B) = (H, A)$, where $U = \varnothing \cup A$,

$$H(o) = \begin{cases} F(o), & o \in A - B \\ G(o), & o \in A \cap B \end{cases}$$

Example 3.2. Let $E = \{e_1, e_2, e_3, e_4\}$ be the parameter set $A = \{e_1, e_2\}$ and $B = \{e_3, e_4, e_5\}$ be the subsets of $E$ and $U = \{h_1, h_2, h_3, h_4, h_5\}$ be the initial universe set. Assume that $(F, A)$ and $(G, B)$ are the soft sets over $U$ defined as following:

$$F(o) = \begin{cases} e_1, & o = e_1 \\ e_2, & o = e_2 \\ e_3, & o = e_3 \\ e_4, & o = e_4 \\ e_5, & o = e_5 \end{cases}$$

$$G(o) = \begin{cases} e_1, & o = e_1 \\ e_2, & o = e_2 \\ e_3, & o = e_3 \\ e_4, & o = e_4 \\ e_5, & o = e_5 \end{cases}$$

Then $(F, A) \cup (G, B)$ is the soft set $(H, A)$, where $H = (H, A)$, and $H(o)$ is given by

$$H(o) = \begin{cases} e_1, & o = e_1 \\ e_2, & o = e_2 \\ e_3, & o = e_3 \\ e_4, & o = e_4 \\ e_5, & o = e_5 \end{cases}$$
Thus, $\mathcal{H}(o) = \mathcal{F}(o)$, $o \in \mathcal{O}$.

Let $\mathcal{L}(o) = \mathcal{F}(o \cup \mathcal{O}(o))$, $o \in \mathcal{O}$.

Thus,
\[
\mathcal{F}(o), \quad o \in \mathcal{O} \rightarrow \mathcal{O}(o) \to \mathcal{O}.
\]

It is seen that $\mathcal{(1)} = \mathcal{(2)}$.

[3] $\mathcal{(F, \mathcal{U}) \sim (\mathcal{Z}, \mathcal{C}) \sim (\mathcal{H}, \mathcal{C}) \sim (\mathcal{F}, \mathcal{U}) \sim (\mathcal{Z}, \mathcal{C})}$

Proof: Let $\mathcal{(F, \mathcal{U})}$ have the following:
\[
\mathcal{F}(o), \quad o \in \mathcal{O} \rightarrow \mathcal{O}(o) \to \mathcal{O}.
\]

Thus,
\[
\mathcal{F}(o), \quad o \in \mathcal{O} \rightarrow \mathcal{O}(o) \to \mathcal{O}.
\]

Here let handle $o \in \mathcal{O}$ in the second equation of the first line. Since $\mathcal{O} \sim \mathcal{C}$,
That is, for the soft sets whose parameter sets are not the same, the operation ∼ has not associativity property on the set $S_{A}(U)$. 

4) $(F, U) ∼ (Z, U) ∼ (Z, U) ∼ (F, U)$

Proof: Let $(F, U) ∼ (Z, U) = (H, U)$, then $∀o ∊ U$;

$$
\begin{align*}
F(o), & \quad o ∊ U ∩ Z' \\
F(o)∪Z(o), & \quad o ∊ U ∩ U = U
\end{align*}
$$

Let $(G, U) ∼ (F, U) = (T, U)$. Then $∀o ∊ U$;

$$
\begin{align*}
Z(o) , & \quad o ∊ U ∩ T' \\
Z(o)∪U F(o) , & \quad o ∊ U ∩ U = U
\end{align*}
$$

That is to say, the operation ∼ has commutativity property in the set $S_{A}(U)$, where the parameter sets of the soft sets are the same.

5) $(F, U) ∼ (F, U) = (F, U)$

Proof: Let $(F, U) ∼ (F, U) = (H, U)$, where $∀o ∊ U$;

$$
\begin{align*}
F'(o), & \quad o ∊ U ∩ U = U \\
F'(o)∪F(o) , & \quad o ∊ U ∩ U = U
\end{align*}
$$

Here $∀o ∊ U; H(o) = F'(o)∪F(o) = F(o)$, thus $(H, U) = (F, U)$.

That is, the operation ∼ has idempotency property on the set $S_{A}(U)$.

6) $(F, U) ∼ ∅_{F} = (F, U) ∼ (F, U)$

Proof: Let $S = (S, U)$. Then, $∀o ∊ U; S(o) = ∅$. Let $(F, U) ∼ (S, U) = (H, U)$, where $∀o ∊ U$;

$$
\begin{align*}
F'(o) , & \quad o ∊ U ∩ U = U \\
F'(o)∪S(o), & \quad o ∊ U ∩ U = U
\end{align*}
$$

Hence, $∀o ∊ U; H(o) = F'(o)∪S(o) = F(o)$, thus $(H, U) = (F, U)$.

Note that, for the soft sets whose parameter set is $\emptyset$, $\emptyset_{F}$ is the identity element for the operation ∼ in $S_{A}(U)$.

REMARK 1: By Theorem 3.3 (1), (2), (4) and (6), $(S_{A}(U), ⊆)$ is a commutative monoid.

7) $(F, U) ∼ ∅_{F} = (F, U)$

Proof: Let $∅_{F} = (S, E)$. Hence $∀o ∊ E; S(o) = ∅$. Let $(F, U) ∼ (S, E) = (H, U)$, then $∀o ∊ U$;

$$
\begin{align*}
F'(o) , & \quad o ∊ U ∩ E = U \\
F'(o)∪S(o), & \quad o ∊ U ∩ E = U
\end{align*}
$$

Hence, $∀o ∊ U; H(o) = F'(o)∪S(o) = F'(o)∪U = F(o)$, thus $(H, U) = (F, U)$.

Note that, for the soft sets (no matter what the parameter set is), $∅_{F}$ is the right identity element for the operation ∼ in $S_{A}(U)$.

8) $(F, U) ~ U_{0} ∼ (F, U) = U_{0}$

Proof: Let $U_{0} = (T, E)$. Then, $∀o ∊ U; T(o) = U$. Let $(F, U) ∼ (T, E) = (H, U)$, where $∀o ∊ U$;

$$
\begin{align*}
F'(o) , & \quad o ∊ U ∩ T' = U \\
F'(o)∪T(o), & \quad o ∊ U ∩ T' = U
\end{align*}
$$

Thus, $∀o ∊ U; H(o) = F'(o)∪T(o) = F'(o)∪U = F(o)∪U$. Hence $(H, U) = (F, U)$.

Note that, for the soft sets whose parameter set is $\emptyset$, $U_{0}$ is the absorbing element for the operation ∼ in $S_{A}(U)$.

9) $(F, U) ∼ U_{0} = U_{0}$

Proof: Let $U_{0} = (T, E)$. Hence, $∀o ∊ E; T(o) = U$. Let $(F, U) ∼ (T, E) = (H, U)$, then $∀o ∊ U$;

$$
\begin{align*}
F'(o) , & \quad o ∊ U ∩ E = U \\
F'(o)∪T(o), & \quad o ∊ U ∩ E = U
\end{align*}
$$

Hence, $∀o ∊ U; H(o) = F'(o)∪T(o) = F'(o)∪U = F(o)∪U$. Hence $(H, U) = (F, U)$.

10) $(F, U) ∼ U_{0} = U_{0}$

Proof: Let $U_{0} = (T, E)$. Then, $∀o ∊ E; T(o) = U$. Let $(F, E) ∼ (T, E) = (H, U)$, where $∀o ∊ U$;

$$
\begin{align*}
T'(o) , & \quad o ∊ E ∩ E = E \\
T'(o)∪F(o), & \quad o ∊ E ∩ E = E
\end{align*}
$$

Hence, $∀o ∊ U; H(o) = T'(o)∪F(o) = U∪F(o) = U$, thus $(H, U) = U_{0}$.

11) $(F, U) ∼ (F, U) = (F, U) ∼ (F, U) = U_{0}$

Proof: Let $U_{0} = (T, E)$. Then, $∀o ∊ E; T(o) = U$. Let $(F, E) ∼ (T, E) = (H, U)$, where $∀o ∊ U$;

$$
\begin{align*}
T'(o) , & \quad o ∊ E ∩ E = E \\
T'(o)∪F(o), & \quad o ∊ E ∩ E = E
\end{align*}
$$

Hence, $∀o ∊ U; H(o) = T'(o)∪F(o) = U∪F(o) = U$, thus $(H, U) = U_{0}$.
**Proof:** Let $(F, U) = (H, U)$, hence $\forall o \in U,$ $H(o) = F(o)$, let $(F, U) \sim \sim (H, U)$ where $\forall o \in U,$

$$T(o) = \begin{cases} F(o), & o \in 0 - C \\ F(o) \cup Z(o), & o \in 0 \cap C \end{cases}$$

Hence, $\forall o \in U, T(o) = F(o) \cup Z(o) = F(o) \cup U = U$, thus $(T, U) = U$

12) $[(F, U) \sim (Z, C)] = (F, U) \sim (H, U)$

**Proof:** Let $(F, U) \sim (Z, C) = (H, U)$. Then, $\forall o \in U,$

$$T(o) = \begin{cases} F(o), & o \in 0 - C \\ F(o) \cup Z(o), & o \in 0 \cap C \end{cases}$$

Thus, $(T, U) = (F, U) \sim (Z, C)$.

In classical theory, $A \cup B = \emptyset \Rightarrow A = \emptyset$ and $B = \emptyset$. Now, we have the following:

13) $(F, U) \sim (Z, U) = \emptyset_0 \Rightarrow (F, U) = \emptyset_0$ and $(Z, U) = \emptyset_0$. Now, we have the following:

**Proof:** Let $(F, U) \sim (Z, U) = (T, U)$. Hence, $\forall o \in U,$

$$T(o) = \begin{cases} F(o), & o \in 0 - C \\ F(o) \cup Z(o), & o \in 0 \cap C \end{cases}$$

Since $(T, U) = (F, U) \sim (Z, U) = \emptyset_0$, then $\forall o \in U, T(o) = F(o) \cup Z(o) = \emptyset_0 \Rightarrow (T, U) = (F, U) \sim (Z, U)$.

In classical theory, for all $A, B \subseteq A$. Now, we have the following:

14) $\emptyset_0 \subseteq (F, U) \sim (Z, U)$ and $\emptyset_0 \subseteq (F, U) \sim (Z, U)$.

In classical theory, for all $A, B \subseteq U$. Now, we have the following:

15) $(F, U) \subseteq (Z, U)$ and $(Z, U) \subseteq (F, U)$

16) $(F, U) \subseteq (Z, U)$ and $(Z, U) \subseteq (F, U) \sim (Z, U)$.

**Proof:** Let $(F, U) \sim (Z, U) = (H, U)$. First of all, $U \subseteq \emptyset$. Moreover, $\forall o \in U,$

$$H(o) = \begin{cases} F(o), & o \in 0 - C \\ F(o) \cup Z(o), & o \in 0 \cap C \end{cases}$$

Since $\forall o \in U, H(o) = F(o) \cup Z(o)$ and $Z(o) \subseteq H(k_0) = F(o) \cup Z(o)$, the proof is completed.

In classical theory, $A \subseteq B \Rightarrow A \cup B = B$. Now, we have the following analogy:

17) $(F, U) \subseteq (Z, U) \Rightarrow (F, U) \sim (Z, U) = (Z, U)$. Now, we have the following:

**Proof:** Let $(F, U) \subseteq (Z, U)$, then $\forall o \in U, F(o) \subseteq G(o)$ and let $(F, U) \sim (Z, U) = (H, U)$. Then, $\forall o \in U,$

$$H(o) = \begin{cases} F(o), & o \in 0 - C \\ F(o) \cup Z(o), & o \in 0 \cap C \end{cases}$$

Since $\forall o \in U, H(o) = F(o) \cup Z(o)$, then $(U, H) = (F, U)$. For the converse, let $(F, U) \sim (Z, U) = (H, U)$. Hence, $(F, U) \sim (Z, U) = (F, U) \sim (Z, U)$ by (16).

18) $(F, U) \subseteq (Z, U) \subseteq (F, U) \sim (Z, U)$

**Proof:** Let $(F, U) \sim (Z, U) = (T, U)$. Hence, $\forall o \in U,$

$$T(o) = \begin{cases} F(o), & o \in 0 - C \\ F(o) \cup Z(o), & o \in 0 \cap C \end{cases}$$

Since for all $\forall o \in U, F(o) \cap Z(o) \subseteq F(o) \cap U$, the rest of the proof is obvious.

In classical theory, $A \cap B = A \cup B \Rightarrow A = B$. Now, we have the following analogy:

19) $(F, U) \sim (Z, U) = (F, U) \sim (Z, U) = (F, U) = (Z, U)$.

**Proof:** Let $(F, U) \sim (G, U) = (F, U) \sim (G, U)$ and $(F, U) \sim (G, U) = (T, U)$, where $\forall o \in U,$

$$W(o) = \begin{cases} F(o), & o \in 0 - C \\ F(o) \cup Z(o), & o \in 0 \cap C \end{cases}$$

Since $(T, U) = (W, U)$, for all $\forall o \in U, T(o) = W(o)$. Hence, $F(o) \cap Z(o) = F(o) \cap U$, so $\forall o \in U, F(o) \cap Z(o) = F(o) \cap U$, that is to say, $(F, U) = (Z, U)$. For the converse, let $(F, U) = (Z, U)$.

Hence, $(F, U) = (Z, U)$. Therefore, $(F, U) = (F, U) \sim (Z, U) = (F, U) \sim (Z, U) = (F, U) \sim (Z, U)$ (by Theorem 3.3 (5), Sezgin et al., 2023a) and $(F, U) \sim (Z, U) = (F, U) \sim (F, U) = (F, U)$ (by Theorem 3.3 (5)). This completes the proof.
In classical set theory, if \( A \subseteq B \) and \( C \subseteq D \), then \( A \cup C \subseteq A \cup D \). Now, we have the following analogy:

20) If \((F,Ü)\subseteq(Z,Ü)\) and \((H,Ü)\subseteq(T,Ü)\), then \((F,Ü)\sim(U,Ü)\subseteq(Z,Ü)\sim(T,Ü)\).

**Proof:** Let \((F,Ü)\subseteq(Z,Ü)\) and \((H,Ü)\subseteq(T,Ü)\). Then, \(\forall o\inÜ\), \(F(o)\subseteq G(o)\) and \(H(o)\subseteq T(o)\). Assume that \((F,Ü)\sim(U,Ü)\subseteq(K,Ü)\) where \(\forall o\inÜ\),

\[
F'(o), \quad o\inÜ-\emptyset\neq\emptyset
\]

\[
K(o)=\left\{ F(o)\cup U(o), \quad o\inÜ\cup\emptyset=\emptyset \right. \]

Let \((Z,Ü)\sim(T,Ü)\subseteq(S,Ü)\) where \(\forall o\inÜ\),

\[
Z'(o), \quad o\inÜ\neq\emptyset
\]

\[
S(o)=\left\{ Z(o)\cup U(o), \quad o\inÜ\cup\emptyset=\emptyset \right. \]

Since, \(\forall o\inÜ\), \(F(o)\subseteq G(o)\) and \(H(o)\subseteq T(o)\), \(K(o)=\left\{ F(o)\cup U(o), \quad o\inÜ\cup\emptyset=\emptyset \right.\)

Hence,

4. **Distribution Rules**

In this section, the distribution of complementary soft binary piecewise union \((U)\) operations over other soft set operations, including complementary extended soft set operations, complementary soft binary piecewise operations, soft binary piecewise operations, and restricted soft set operations, is thoroughly examined. Several intriguing findings are as a result.

4.1 Distribution of complementary soft binary piecewise union \((U)\) operation over extended soft set operations:

1) Left-distribution of complementary soft binary piecewise union \((U)\) operation over extended soft set operations:

\[
(Z,Ü)\subseteq(H,Ü) \Rightarrow [(Z,Ü)\cap(H,Ü)] = [(Z,Ü)\cap U] \subseteq [(H,Ü)\sim(U,Ü)]\]

**Proof:** Let first handle the left hand side of the equality and let \((Z,Ü)\cap(H,Ü)\subseteq(M,Ü)\) where \(\forall o\inÜ\),

\[
Z(o), \quad o\inÜ\subseteq\emptyset
\]

\[
M(o)=\left\{ Z(o)\cup H(o), \quad o\inÜ\subseteq\emptyset \right. \]

Assume that \((F,Ü)\sim(M,Ü)\subseteq(N,Ü)\) where \(\forall o\inÜ\),

\[
F'(o), \quad o\inÜ-\emptyset\neq\emptyset
\]

\[
N(o)=\left\{ F(o)\cup M(o), \quad o\inÜ-\emptyset\subseteq\emptyset \right. \]

Hence,

\[
F'(o), \quad o\inÜ-(Ü\cup\emptyset)\neq\emptyset
\]

\[
N(o)=\left\{ F(o)\cup M(o), \quad o\inÜ-\emptyset\subseteq\emptyset \right. \]

\[
F(o)\cup M(o), \quad o\inÜ-\emptyset\subseteq\emptyset
\]

Now let handle the right hand side of the equality: \( ([F(o)] \cup (H,C)) \cup_\cup (V,U), \) where \( \forall o \in U \):

- \( V(o) = F'(o), \quad o \in U \cap C \)
- \( V(o) = F(o) \cup H(o), \quad o \in U \cap C \)
- \( V(o) = F(o) \cup H(o), \quad o \in U \cap C \)
- \( V(o) = F'(o), \quad o \in U \cap C \)

Now let handle the right hand side of the equality: \( ([F(o)] \cup (H,C)) \cup_\cup (V,U), \) where \( \forall o \in U \):

- \( V(o) = F'(o), \quad o \in U \cap C \)
- \( V(o) = F(o) \cup H(o), \quad o \in U \cap C \)
- \( V(o) = F(o) \cup H(o), \quad o \in U \cap C \)
- \( V(o) = F'(o), \quad o \in U \cap C \)

Suppose that \( (Z(o) \cup H(o)) \cup (V,U), \) where \( \forall o \in U \):

- \( V(o) = F'(o), \quad o \in U \cap C \)
- \( V(o) = F(o) \cup H(o), \quad o \in U \cap C \)
- \( V(o) = F(o) \cup H(o), \quad o \in U \cap C \)
- \( V(o) = F'(o), \quad o \in U \cap C \)

Let \( (F,U) \cup (V,U), \) where \( \forall o \in U \):

- \( V(o) = F'(o), \quad o \in U \cap C \)
- \( V(o) = F(o) \cup H(o), \quad o \in U \cap C \)
- \( V(o) = F(o) \cup H(o), \quad o \in U \cap C \)
- \( V(o) = F'(o), \quad o \in U \cap C \)

Hence, \( (F,U) \cup (V,U), \) where \( \forall o \in U \):

- \( V(o) = F'(o), \quad o \in U \cap C \)
- \( V(o) = F(o) \cup H(o), \quad o \in U \cap C \)
- \( V(o) = F(o) \cup H(o), \quad o \in U \cap C \)
- \( V(o) = F'(o), \quad o \in U \cap C \)

It is seen that \( (1)=(2). \)

ii) Right-distribution of complementary soft binary piecewise union (U) operation over extended soft set operations:

1) \( ([F,U] \cup (Z,C)) \cup_\cup (H,C) = ([F,U] \cup_\cup (Z,C)) \cup_\cup (H,C) \) where \( \forall o \in U \cap C = \emptyset \)

Proof: Let first handle the left hand side of the equality and let \( (F,U) \cup (Z,C) = (M, U \cup C), \) where \( \forall o \in U \cup C \):

- \( M(o) = Z(o), \quad o \in U \cup C \)
- \( M(o) = Z(o), \quad o \in U \cup C \)
- \( M(o) = Z(o), \quad o \in U \cup C \)
- \( M(o) = Z(o), \quad o \in U \cup C \)

Hence, \( (F,U) \cup (Z,C) = (M, U \cup C), \) where \( \forall o \in U \cup C \):

- \( F(o) = F'(o), \quad o \in U \cup C \)
- \( F(o) = F'(o), \quad o \in U \cup C \)
- \( F(o) = F'(o), \quad o \in U \cup C \)
- \( F(o) = F'(o), \quad o \in U \cup C \)

Now let handle the right hand side of the equality: \( ([F,U] \cup (H,C)) \cup_\cup (V,U), \) where \( \forall o \in U \):

- \( V(o) = F'(o), \quad o \in U \cap C \)
- \( V(o) = F(o) \cup H(o), \quad o \in U \cap C \)
- \( V(o) = F(o) \cup H(o), \quad o \in U \cap C \)
- \( V(o) = F'(o), \quad o \in U \cap C \)

It is seen that \( (1)=(2). \)

2) \( ([F,U] \cup_\cup (Z,C)) \cup_\cup (H,C) = ([F,U] \cup_\cup (Z,C)) \cup_\cup (H,C) \) where \( \forall o \in U \cap C = \emptyset \)

Proof: Let first handle the left hand side of the equality and let \( (F,U) \cup (Z,C) = (M, U \cup C), \) where \( \forall o \in U \cup C \):

- \( M(o) = Z(o), \quad o \in U \cup C \)
- \( M(o) = Z(o), \quad o \in U \cup C \)
- \( M(o) = Z(o), \quad o \in U \cup C \)
- \( M(o) = Z(o), \quad o \in U \cup C \)

Hence, \( (F,U) \cup (Z,C) = (M, U \cup C), \) where \( \forall o \in U \cup C \):

- \( F(o) = F'(o), \quad o \in U \cup C \)
- \( F(o) = F'(o), \quad o \in U \cup C \)
- \( F(o) = F'(o), \quad o \in U \cup C \)
- \( F(o) = F'(o), \quad o \in U \cup C \)
Thus, \( N(o) = H'(o) \), \( o \in C \)  

Let \((Z,C) = (W,C)\), where \( \forall o \in C \);  

\[
W(o) = \begin{cases} 
Z(o), & o \in C \\
Z(o) \cup H(o), & o \in C 
\end{cases}
\]

Assume that \((V,U) \cap (W,C) = (T,U \cup C)\), where \( \forall o \in U \cup C \);  

\[
T(o) = \begin{cases} 
V(o), & o \in U \\
W(o), & o \in C 
\end{cases}
\]

Hence,  

\[
F'(o), \quad o \in U-C \\
F(o) \cup H(o), \quad o \in U-C \\
Z(o), \quad o \in U-C \\
Z(o) \cup H(o), \quad o \in U-C 
\]

Now let handle the right hand side of the equality \( \begin{cases} 
F'(o), & o \in U-C \\
F(o) \cup H(o), & o \in U-C 
\end{cases} \).  

Assume that \((Z,C) = (W,U)\), where \( \forall o \in U \);  

\[
V(o) = \begin{cases} 
F'(o), & o \in U-C \\
F(o) \cup H(o), & o \in U-C 
\end{cases}
\]

Let \((H,C) = (T,U)\), where \( \forall o \in U \);  

\[
W(o) = \begin{cases} 
H(o), & o \in C \\
H(o) \cup F(o), & o \in C 
\end{cases}
\]

Suppose that \((V,U) \cap (W,U) = (T,U \cup C)\), where \( \forall o \in U \);  

\[
T(o) = \begin{cases} 
V(o), & o \in U \\
W(o), & o \in C 
\end{cases}
\]

It is seen that \( 1 \equiv 2 \).  

4) \( \begin{cases} 
(F,U) \lambda_s (Z,C) \equiv (H,U) = ([F,U] \cap (Z,C)) \cup ([Z,C] \cap (H,U)) \cup (F,U) \\
\forall o \in U \cup C = \phi 
\end{cases} \)  

**Proof:** Let first handle the left hand side of the equality. Assume \( (Z,C) \equiv (H,U) = (M, C \cup U) \), where \( \forall o \in C \cup U \);  

\[
M(o) = \begin{cases} 
Z(o), & o \in C-C \\
H'(o), & o \in C-C \\
Z(o) \cup H(o), & o \in C-C 
\end{cases}
\]

Let \((F,U) = (M, C \cup U) = (N,U)\), where \( \forall o \in U \);  

\[
N(o) = \begin{cases} 
F'(o), & o \in U-C \\
F(o) \cup M(o), & o \in U-C 
\end{cases}
\]

Thus,  

\[
N(o) = \begin{cases} 
F'(o), & o \in U-C \cup C-C = U \cap C \cap C \\
F(o) \cup Z(o), & o \in U-C \cup C \\
F(o) \cup H'(o), & o \in U-C \cup C \\
F(o) \cup (Z(o) \cup H(o)), & o \in U \cap C \cap C = U \cap C \cap C 
\end{cases}
\]
Now let handle the right hand side of the equality: \([F,\emptyset] \sim (Z,\emptyset) \cup \{F,\emptyset\}\) where \(\emptyset \neq \emptyset\).

Thus, \(W(o) = \{F(o)\cup H(o), o\in(\emptyset,\emptyset)\}, o\in\emptyset\}\)

Thus, \(W(o) = \{F(o)\cup H(o), o\in(\emptyset,\emptyset)\}, o\in\emptyset\}\)

Thus, \(W(o) = \{F(o)\cup H(o), o\in(\emptyset,\emptyset)\}, o\in\emptyset\}\)

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Thus, \(W(o) = \{F(o)\cup H(o), o\in(\emptyset,\emptyset)\}, o\in\emptyset\}\)

Thus, \(W(o) = \{F(o)\cup H(o), o\in(\emptyset,\emptyset)\}, o\in\emptyset\}\)

Thus, \(W(o) = \{F(o)\cup H(o), o\in(\emptyset,\emptyset)\}, o\in\emptyset\}\)

Thus, \(W(o) = \{F(o)\cup H(o), o\in(\emptyset,\emptyset)\}, o\in\emptyset\}\)

Thus, \(W(o) = \{F(o)\cup H(o), o\in(\emptyset,\emptyset)\}, o\in\emptyset\}\)

Thus, \(W(o) = \{F(o)\cup H(o), o\in(\emptyset,\emptyset)\}, o\in\emptyset\}\)

Thus, \(W(o) = \{F(o)\cup H(o), o\in(\emptyset,\emptyset)\}, o\in\emptyset\}\)

Thus, \(W(o) = \{F(o)\cup H(o), o\in(\emptyset,\emptyset)\}, o\in\emptyset\}\)

Thus, \(W(o) = \{F(o)\cup H(o), o\in(\emptyset,\emptyset)\}, o\in\emptyset\}\)

Thus, \(W(o) = \{F(o)\cup H(o), o\in(\emptyset,\emptyset)\}, o\in\emptyset\}\)

Thus, \(W(o) = \{F(o)\cup H(o), o\in(\emptyset,\emptyset)\}, o\in\emptyset\}\)

Thus, \(W(o) = \{F(o)\cup H(o), o\in(\emptyset,\emptyset)\}, o\in\emptyset\}\)

Thus, \(W(o) = \{F(o)\cup H(o), o\in(\emptyset,\emptyset)\}, o\in\emptyset\}\)

Thus, \(W(o) = \{F(o)\cup H(o), o\in(\emptyset,\emptyset)\}, o\in\emptyset\}\)

Thus, \(W(o) = \{F(o)\cup H(o), o\in(\emptyset,\emptyset)\}, o\in\emptyset\}\)

Thus, \(W(o) = \{F(o)\cup H(o), o\in(\emptyset,\emptyset)\}, o\in\emptyset\}\)

Thus, \(W(o) = \{F(o)\cup H(o), o\in(\emptyset,\emptyset)\}, o\in\emptyset\}\)

Thus, \(W(o) = \{F(o)\cup H(o), o\in(\emptyset,\emptyset)\}, o\in\emptyset\}\)

Thus, \(W(o) = \{F(o)\cup H(o), o\in(\emptyset,\emptyset)\}, o\in\emptyset\}\)

Thus, \(W(o) = \{F(o)\cup H(o), o\in(\emptyset,\emptyset)\}, o\in\emptyset\}\)

Thus, \(W(o) = \{F(o)\cup H(o), o\in(\emptyset,\emptyset)\}, o\in\emptyset\}\)

Thus, \(W(o) = \{F(o)\cup H(o), o\in(\emptyset,\emptyset)\}, o\in\emptyset\}\)

Thus, \(W(o) = \{F(o)\cup H(o), o\in(\emptyset,\emptyset)\}, o\in\emptyset\}\)

Thus, \(W(o) = \{F(o)\cup H(o), o\in(\emptyset,\emptyset)\}, o\in\emptyset\}\)

Thus, \(W(o) = \{F(o)\cup H(o), o\in(\emptyset,\emptyset)\}, o\in\emptyset\}\)

Thus, \(W(o) = \{F(o)\cup H(o), o\in(\emptyset,\emptyset)\}, o\in\emptyset\}\)

Thus, \(W(o) = \{F(o)\cup H(o), o\in(\emptyset,\emptyset)\}, o\in\emptyset\}\)
Assume that $(F, \cup) \perp \sim (H, C) = (V, U)$, where $\forall o \in U$;

$V(o) = \begin{cases} 
F(o), & o \in U \cap C' \\
F'(o) \cup H(o), & o \in U \cap C \end{cases}$

Let $(Z, C) \perp (H, C) = (W, C)$, where $\forall o \in C$;

$W(o) = \begin{cases} 
Z(o), & o \in U \cap C' \\
Z(o) \cup H(o), & o \in U \cap C \end{cases}$

Assume that $(V, U) \cup (W, C) = (T, U \cap C)$, where $\forall o \in U \cap C$;

Thus,

$T(o) = \begin{cases} 
V(o), & o \in U \cap C' \\
W(o), & o \in U \cap C \end{cases}$

$V(o) \cup W(o), & o \in U \cap C$

Now let the right hand side of the equality:

$[(F, \cup) \perp \sim (Z, C)] \cup U$
\( V(o) = \begin{cases} \{F(o), o \in \emptyset\} \cup Z(o), & o \in \emptyset \cap C' \\ F(o) \cup Z(o), & o \in (\emptyset \cap C) \cap (C' \cap \emptyset) = \emptyset \end{cases} \)

Let \((H,C) \sim (F,U) \cap (W,C), \) where \( V \circ o \in C; \)

\( W(o) = \begin{cases} \{F(o) \cup U H(o), o \in \emptyset \} \cup Z(o), & o \in \emptyset \cap C' \\ H(o), & o \in C \cap U \\ H(o) \cup U F(o), & o \in C \cap \emptyset \end{cases} \)

Assume that \((V,U) \cap C = \emptyset \), where \( V \circ o \in C; \)

\( T(o) = \begin{cases} \{V(o), o \in \emptyset \} \cap W(o), & o \in \emptyset \cap C' \\ V(o) \cap W(o), & o \in C \cap \emptyset \end{cases} \)

Therefore,

\( \begin{align*}
T(o) &= F(o), o \in (\emptyset \cap C) \cap (C \cap \emptyset) = \emptyset \\
F(o) \cap U F(o), o \in (C \cap \emptyset) \cap (C \cap \emptyset) = C \cap \emptyset \\
[\{F(o) \cup U F(o), o \in (\emptyset \cap C) \cap (C \cap \emptyset) = \emptyset \} \cup Z'(o), o \in (\emptyset \cap C) \cap (C' \cap \emptyset) = \emptyset] \\
[\{F(o) \cup U F(o), o \in (\emptyset \cap C) \cap (C \cap \emptyset) = \emptyset \} \cup H'(o), o \in (\emptyset \cap C) \cap (C' \cap \emptyset) = \emptyset] \\
[\{F(o) \cup U F(o), o \in (\emptyset \cap C) \cap (C \cap \emptyset) = \emptyset \} \cup Z'(o), o \in (\emptyset \cap C) \cap (C' \cap \emptyset) = \emptyset] \\
[\{F(o) \cup U F(o), o \in (\emptyset \cap C) \cap (C \cap \emptyset) = \emptyset \} \cup H'(o), o \in (\emptyset \cap C) \cap (C' \cap \emptyset) = \emptyset] \\
\end{align*} \)

It is seen that \((1) = (2). \)

**REMARK 2:** In Sezgin and Yavuz (2023a), it is proved that \((S_{o}(U), \tilde{H})\) is a commutative monoid with identity \( U \). And in Remarkt, we show that \((S_{o}(U), \sim)\) is a commutative semiring. Moreover, \( U_{o} \sim (F,U)=U_{o} \). That is to say, \( U_{o} \) is the left-absorbing element for the operation \( \sim \). Besides, by the \( U \)

\( \begin{align*}
\end{align*} \)
Now let handle the right hand side of the equality: \([F,\emptyset] \cup (H,C) \sim (Z,C) \sim (H,C) \cup (Z,C) \sim (H,C) \cup (Z,C) \sim (H,C) \cup (Z,C) \sim (H,C) \cup (Z,C) \sim (H,C)\) \(\sim \)

\(\{Z,C\} \sim (H,C)\)

\(\sim \)

Let \((F,\emptyset) \sim (H,C) = (V,\emptyset)\), where \(\forall o \in \emptyset\);  
\[
F'(o), \quad o \in \emptyset - \emptyset
\]
\[
F(o) \cup H(o), \quad o \in \emptyset - \emptyset
\]
\[
Z'(o), \quad o \in \emptyset - \emptyset
\]
\[
Z(o) \cup H(o), \quad o \in \emptyset - \emptyset
\]

Suppose that \((V,\emptyset) \cup (W,\emptyset) = (T,\emptyset)\), where \(\forall o \in \emptyset\);

\(T(o) = \)

\(V(o), \quad o \in \emptyset - \emptyset
\]
\[
V(o) \cup W(o), \quad o \in \emptyset - \emptyset
\]

Hence,

\[
F'(o), \quad o \in \emptyset - \emptyset
\]
\[
F(o) \cup H(o), \quad o \in \emptyset - \emptyset
\]
\[
Z'(o), \quad o \in \emptyset - \emptyset
\]
\[
Z(o) \cup H(o), \quad o \in \emptyset - \emptyset
\]

It is seen that \((1) = (2)\).

4.4 Distribution of complementary soft binary piecewise union (U) operation over complementary soft binary piecewise operations:

1) Left distribution of complementary soft binary piecewise union (U) operation over complementary soft binary piecewise operations:

\(\{F,\emptyset\} \sim (Z,C) \sim (H,C) = (F,\emptyset) \sim (Z,C) \sim (H,C) \sim (F,\emptyset)\)

where \(\emptyset \cup \emptyset = \emptyset\)

**Proof:** Let first handle the left hand side of the equality, let \(Z,C \sim (H,C)\) where \(\forall \emptyset \in \emptyset\);

\[
\left[ Z(o), \quad o \in \emptyset - \emptyset \right]
\]
\[
Z(o) \cup H(o), \quad o \in \emptyset - \emptyset
\]

Let \((F,\emptyset) \sim (M,\emptyset)\) where \(\forall \emptyset \in \emptyset\);

\[
F'(o), \quad o \in \emptyset - \emptyset
\]
\[
F(o) \cup M(o), \quad o \in \emptyset - \emptyset
\]

Therefore,

\[
F'(o), \quad o \in \emptyset - \emptyset
\]
\[
F(o) \cup M(o), \quad o \in \emptyset - \emptyset
\]

Here let handle the right hand side of the equality, let \(Z,C \sim (H,C)\) where \(\forall \emptyset \in \emptyset\);

\[
\left[ F'(o), \quad o \in \emptyset - \emptyset \right]
\]
\[
F(o) \cup M(o), \quad o \in \emptyset - \emptyset
\]

Thus,

\[
F'(o), \quad o \in \emptyset - \emptyset
\]
\[
F(o) \cup M(o), \quad o \in \emptyset - \emptyset
\]

2) Right distribution of complementary soft binary piecewise union (U) operation over complementary soft binary piecewise operations:

\(\{F,\emptyset\} \sim (Z,C) \sim (H,C) = (F,\emptyset) \sim (Z,C) \sim (H,C) \sim (F,\emptyset)\)

where \(\emptyset \cup \emptyset = \emptyset\)

**Proof:** Let first handle the left hand side of the equality, let \(Z,C \sim (H,C)\) where \(\forall \emptyset \in \emptyset\);

\[
\left[ F'(o), \quad o \in \emptyset - \emptyset \right]
\]
\[
F(o) \cup M(o), \quad o \in \emptyset - \emptyset
\]

Here let handle the right hand side of the equality, let \(Z,C \sim (H,C)\) where \(\forall \emptyset \in \emptyset\);

\[
\left[ F'(o), \quad o \in \emptyset - \emptyset \right]
\]
\[
F(o) \cup M(o), \quad o \in \emptyset - \emptyset
\]

Thus,

\[
F'(o), \quad o \in \emptyset - \emptyset
\]
\[
F(o) \cup M(o), \quad o \in \emptyset - \emptyset
\]

Here let handle the right hand side of the equality, let \(Z,C \sim (H,C)\) where \(\forall \emptyset \in \emptyset\);

\[
\left[ F'(o), \quad o \in \emptyset - \emptyset \right]
\]
\[
F(o) \cup M(o), \quad o \in \emptyset - \emptyset
\]
\[(H,C) = (M,C), \text{ where } o \in C; \]

\[M(o) = \begin{cases} 
Z'(o), & o \in \text{C} \\
Z'(o) \cap H(o), & o \in \text{C} \\
F'(o), & o \in \text{C'} \\
F(o), & o \in (\text{C} \cup \text{C'}) \end{cases}, \]

\[N(o) = \begin{cases} 
F(o) \cup Z(o), & o \in (\text{C} \cup \text{C'}) \\
F(o), & o \in \text{C} \\
Z'(o), & o \in \text{C'} \\
F'(o), & o \in \text{C'} \\
F(o), & o \in \text{C} \cup \text{C'} \\
F'(o), & o \in \text{C'} \end{cases} \]

Therefore,

\[M(o) = \begin{cases} 
F(o), & o \in \text{C} \\
F(o) \cap Z(o), & o \in \text{C} \cup \text{C'} \\
F(o) \cap H(o), & o \in \text{C} \cup \text{C'} \\
F'(o), & o \in \text{C'} \end{cases}, \]

\[N(o) = \begin{cases} 
F(o) \cup Z(o), & o \in \text{C} \cup \text{C'} \\
F(o), & o \in \text{C} \\
Z'(o), & o \in \text{C'} \\
F'(o), & o \in \text{C'} \\
F(o), & o \in \text{C} \cup \text{C'} \\
F'(o), & o \in \text{C'} \end{cases} \]

Now let handle the right hand side of the equality: \([(F,U) \sim (Z,C)] \cap H(C) \]

\[F(o), \quad o \in \text{C} \]

\[F(o) \cup Z(o), \quad o \in \text{C} \cup \text{C'} \]

\[F(o) \cap H(o), \quad o \in \text{C} \cup \text{C'} \]

\[F'(o), \quad o \in \text{C'} \]

\[F(o), \quad o \in \text{C} \cup \text{C'} \]

\[F'(o), \quad o \in \text{C'} \]

\[F(o), \quad o \in \text{C} \cup \text{C'} \]

\[F'(o), \quad o \in \text{C'} \]

\[F(o), \quad o \in \text{C} \cup \text{C'} \]

\[F'(o), \quad o \in \text{C'} \]

\[F(o), \quad o \in \text{C} \cup \text{C'} \]

\[F'(o), \quad o \in \text{C'} \]

\[F(o), \quad o \in \text{C} \cup \text{C'} \]

\[F'(o), \quad o \in \text{C'} \]

\[F(o), \quad o \in \text{C} \cup \text{C'} \]

\[F'(o), \quad o \in \text{C'} \]

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\[F'(o), \quad o \in \text{C'} \]

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\[F'(o), \quad o \in \text{C'} \]

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\[F'(o), \quad o \in \text{C'} \]

\[F(o), \quad o \in \text{C} \cup \text{C'} \]

\[F'(o), \quad o \in \text{C'} \]

\[F(o), \quad o \in \text{C} \cup \text{C'} \]

\[F'(o), \quad o \in \text{C'} \]

\[F(o), \quad o \in \text{C} \cup \text{C'} \]
Proof: Let first handle the left hand side of the equality, let (F,Ü) ~ (Z,Ç)=(M,Ç∩C), where ∀o∊Ü,
\[
\begin{align*}
F(o), & \quad o∊(Ç∩C)
\end{align*}
\]
M(o)=
\[
\begin{align*}
F(o) \cup Z(o), & \quad o∊(Ç∩C)
\end{align*}
\]
Let (M,Ü) ~ (H,C)=(N,Ü), where ∀o∊Ü,
\[
\begin{align*}
M'(o), & \quad o∊(Ç∩C)
\end{align*}
\]
N(o)=
\[
\begin{align*}
M(o)∪H(o), & \quad o∊(Ç∩C)
\end{align*}
\]
Hence,
\[
\begin{align*}
N(o)=
\begin{cases}
F(o), & o∊(Ç∩C) \cup (Ç∩C)'
F(o)\cap Z(o), & o∊(Ç∩C)' \cup (Ç∩C)
\end{cases}
\end{align*}
\]
Thus,
\[
\begin{align*}
N(o)=
\begin{cases}
F(o), & o∊(Ç∩C) \cup (Ç∩C)'
F(o)\cap Z(o), & o∊(Ç∩C)' \cup (Ç∩C)
\end{cases}
\end{align*}
\]
Now let handle the right hand side of the equality: \( [(F,Ü) ~ (Z,Ç)] \cap (Ç∩C') \)
\[
\begin{align*}
F(o), & \quad o∊(Ç∩C) \cup (Ç∩C)'
\end{align*}
\]
\[
\begin{align*}
F(o)\cap Z(o), & \quad o∊(Ç∩C)' \cup (Ç∩C)
\end{align*}
\]
Let (H,C)=(W,Ü), so ∀o∊Ü,
\[
\begin{align*}
W(o)=
\begin{cases}
F(o), & o∊(Ç∩C) \cup (Ç∩C)'
F(o)\cap Z(o), & o∊(Ç∩C)' \cup (Ç∩C)
\end{cases}
\end{align*}
\]
Assume that (Z,Ç) ~ (W,Ç), where ∀o∊Ç;
\[
\begin{align*}
Z(o), & \quad o∊Ç
\end{align*}
\]
W(o)=
\[
\begin{align*}
Z(o)∪H(o), & \quad o∊Ç∩C
\end{align*}
\]
Let (V,Ü) ~ (W,Ç)=(T,Ü), where ∀o∊Ü;
\[
\begin{align*}
V(o)=
\begin{cases}
F(o), & o∊(Ç∩C) \cup (Ç∩C)'
F(o)\cap Z(o), & o∊(Ç∩C)' \cup (Ç∩C)
\end{cases}
\end{align*}
\]
T(o)=
\[
\begin{align*}
V(o)∪W(o), & \quad o∊Ç∩C
\end{align*}
\]
Therefore,
\[
\begin{align*}
F(o), & \quad o∊(Ç∩C) \cup (Ç∩C)'
F(o)∪H(o), & \quad o∊(Ç∩C) \cup (Ç∩C)'
T(o)=
\begin{cases}
F(o)∪Z(o), & o∊(Ç∩C)' \cup (Ç∩C)
F(o)∪Z(o)∪H(o), & o∊(Ç∩C)' \cup (Ç∩C)
\end{cases}
\end{align*}
\]
\[
\begin{align*}
[F'(o)∪H(o)]∪Z(o), & \quad o∊(Ç∩C)' \cup (Ç∩C)
[F'(o)∪H(o)]∪Z(o)∪H(o), & \quad o∊(Ç∩C)' \cup (Ç∩C)
\end{align*}
\]
It is seen that (1)=(2).

4.5 Distribution of complementary soft binary piecewise union (U) operation over restricted soft set operations:
The followings are hold where Ç∩C ≠ ø and o∊Ç∩C ∪ C = ø.
\[ F'(o), \quad o \in U-(\cap C) \]
\[ N(o)=\begin{cases} F'(o), \quad o \in U-(\cap C) \\ F(o) \cup M(o), \quad o \in U-(\cap C) \end{cases} \]

Thus,
\[ F'(o), \quad o \in U-(\cap C) \]
\[ N(o)=\begin{cases} F'(o) \cup [Z(o) \cap H(o)], \quad o \in U-(\cap C) \\ F(o) \cup [Z(o) \cap H(o)], \quad o \in U-(\cap C) \end{cases} \]

Now let handle the right hand side of the equality, \( [(F,\cap U) \cap (Z,\cup C)] \cap \theta \]
\[ (F,\cap U) \cap (H,\cup C). \]

**Proof:** Let first handle the left hand side of the equality, suppose \((Z,\cup C) \cap (H,\cap C) \neq \emptyset \) and so \( \forall o \in C \cap U, M(o) \cap Z(o) \cap H(o). \) Let \((F,\cap U) \sim U (M,\cap C) \cap (N,\cup U), \) so \( \forall o \in U, \)
\[ F(o), \quad o \in U-(\cap C) \]
\[ N(o)=\begin{cases} F(o), \quad o \in U-(\cap C) \\ F(o) \cup M(o), \quad o \in U-(\cap C) \end{cases} \]

Thus,
\[ F(o), \quad o \in U-(\cap C) \]
\[ N(o)=\begin{cases} F(o) \cup [Z(o) \cap H(o)], \quad o \in U-(\cap C) \\ F(o) \cup [Z(o) \cap H(o)], \quad o \in U-(\cap C) \end{cases} \]
Let \((F, U) \sim (H, C)\), so \(V \circ U\). Then,

\[
W(o) = \begin{cases} 
F'(o), & o \in \bar{U} - C \\
F(o), & o \in \bar{U} \cap C
\end{cases}
\]

Assume that \((V, U) \cup (W, U) = (T, U)\), and so \(V \circ U\), \(T(o) = V(o) \cap W(o)\).

\[
T(o) = \begin{cases} 
F'(o), & o \in \bar{U} - C \\
F(o), & o \in \bar{U} \cap C
\end{cases}
\]

\[
F'(o), & o \in \bar{U} \cap C
\]

\[
F(o), & o \in \bar{U} - C
\]

\[
F'(o), & o \in \bar{U} \cap C
\]

\[
F(o), & o \in \bar{U} - C
\]

\[
F'(o), & o \in \bar{U} \cap C
\]

\[
F(o), & o \in \bar{U} - C
\]

\[
F'(o), & o \in \bar{U} \cap C
\]

\[
F(o), & o \in \bar{U} - C
\]

\[
F'(o), & o \in \bar{U} \cap C
\]

\[
F(o), & o \in \bar{U} - C
\]

\[
F'(o), & o \in \bar{U} \cap C
\]

\[
F(o), & o \in \bar{U} - C
\]

\[
F'(o), & o \in \bar{U} \cap C
\]

5. Conclusion

By defining a new type of soft set operation, which we refer to as complementary soft binary piecewise union operation, we hope to add to the literature on soft sets. Investigated are the operations’ fundamental algebraic characteristics. The relationships between this new soft set operation and other soft set operations, such as extended soft set operations, complementary extended soft set operations, soft binary piecewise operations, complementary soft binary piecewise operations, and restricted soft set operations, are also discovered by looking at the distribution rules. Besides, we demonstrate that the set of all the soft sets with a fixed parameter set together with the complementary soft binary piecewise union operation and the soft binary piecewise intersection operation is a zero-symmetric near-semiring. This article can be seen as a theoretical investigation into soft sets. In the future, research into the distribution of more soft set operations over complementary soft binary piecewise union operations may be conducted, and new kinds of soft set operations may be defined.

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